Lesson 4

Thermomechanical Measurements for Energy Systems (MENR)

Measurements for Mechanical Systems and Production (MMER)

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• RAPIDITY (Dynamic Response)

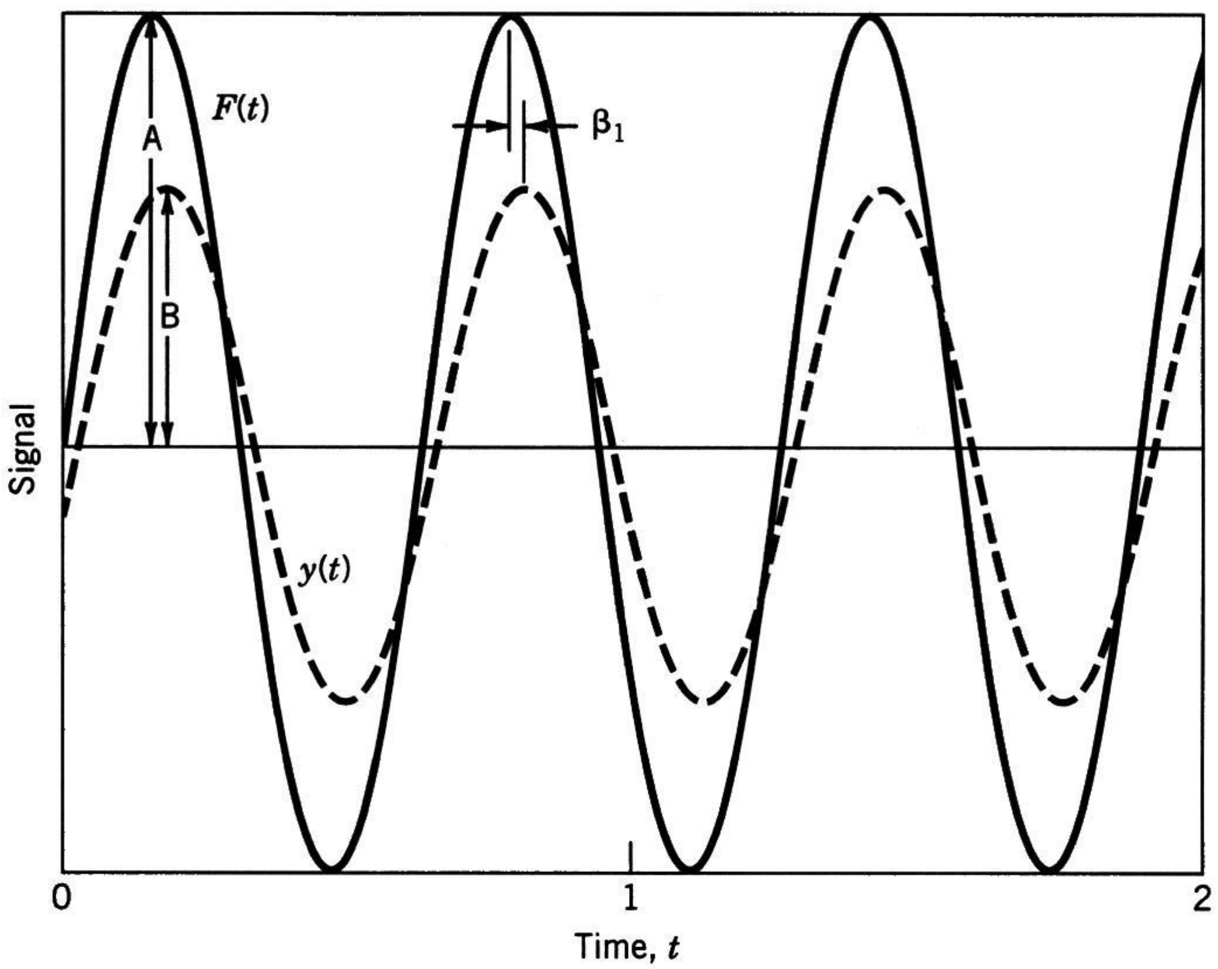
So far the ***measurand*** (the *physical quantity* for which we wish to *measure the intensity*) has been considered strictly CONSTANT during the whole measurement procedure !
From here on it will chance its intensity with time during the measurement … the *measurand* becomes a *function of time !*

both the ***input measurand*** and the ***output measurement*** are functions of time → ***i(t)*** ***u(t)***

We may call ***rapidity of an instrument*** "the attitude to *correctly follow* the *changes of the measurand* during time"

Rapidity of *mechanical instruments* is always limited by the ***inertia*** and the ***damping effects*** of its moving parts !
Rapidity for *electronic instruments* is always limited by the combination of its ***capacitive*** and ***inductive reactances*** !

An instrument with *insufficient rapidity* (***insufficient dynamic response***) during a measurement will output a *measurement waveform* which will be attenuated and out of phase (delayed) with respect to the measurand !
The *output measurement waveform* will be ***distorted*** with respect to the *input measurand waveform* …



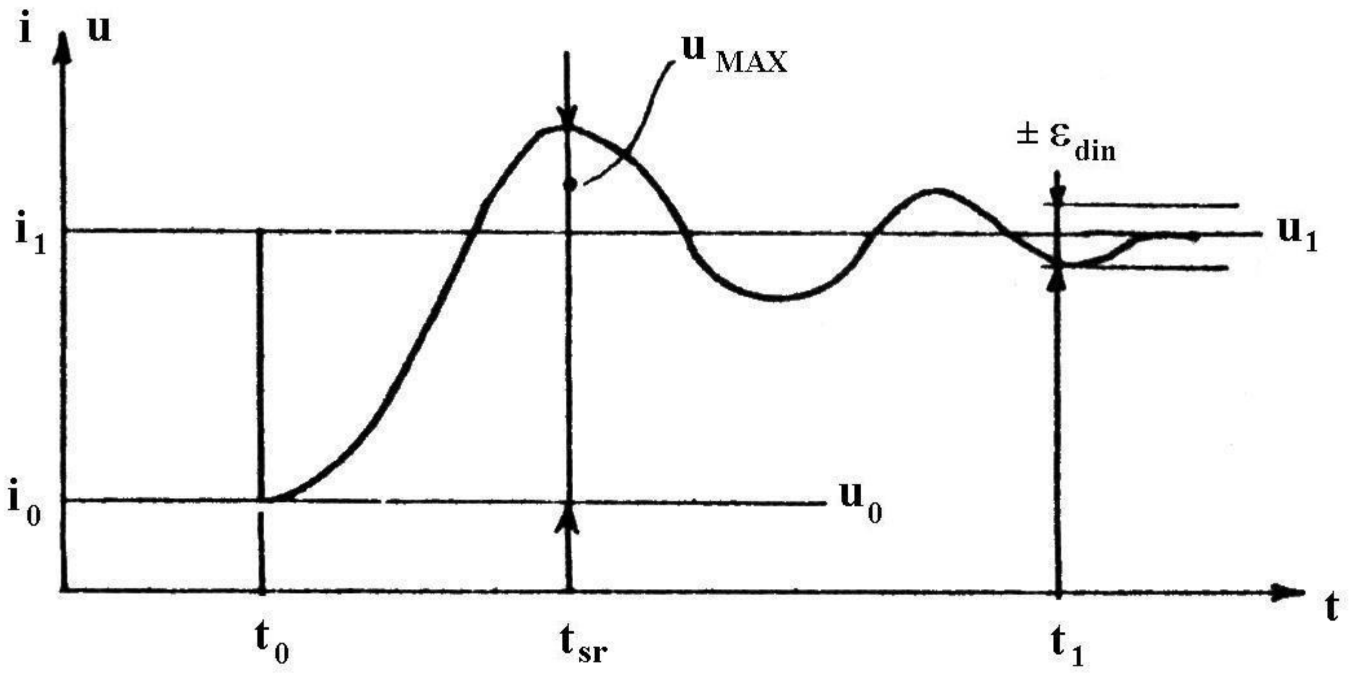
Relationship between a sinusoidal input and output: amplitude, frequency, and time lag.

We have basically two schemes or methods to study the dynamic response of a measurement instrument:

1. **STEP RESPONSE** with which we can get some important parameters such the *settling time* and the *time constant*
2. **FREQUENCY RESPONSE** with which we can get other important parameters such the *cut-off frequency* and the *damping factor* …

There is also a third scheme, which is used less often: the **ramp response** that leads to the *delay time* …

Step Response



At time t_0 the input signal changes instantaneously its value from i_0 to i_1

The instrument output will try to follow this change, form u_0 *to* u_1 , the best he can …

However, it will take some time $\boldsymbol{t_s = t_1 - t_0}$ to reach the correct final value within a certain error $\boldsymbol{\pm\varepsilon_d}$!

$\boldsymbol{\pm\varepsilon_d}$ is the ***dynamic error,*** which has to be established in advance by the operator …

$\boldsymbol{t_s = t_1 - t_0}$ is the ***settling time*** of the instrument, which provides a first idea of its dynamic response

$\boldsymbol{t_{SLEW} = t_{sr} - t_0}$ is the ***slew rate,*** which is the time the instrument takes to reach the first overshoot peak …

… not all the instruments show an overshoot during the step response !

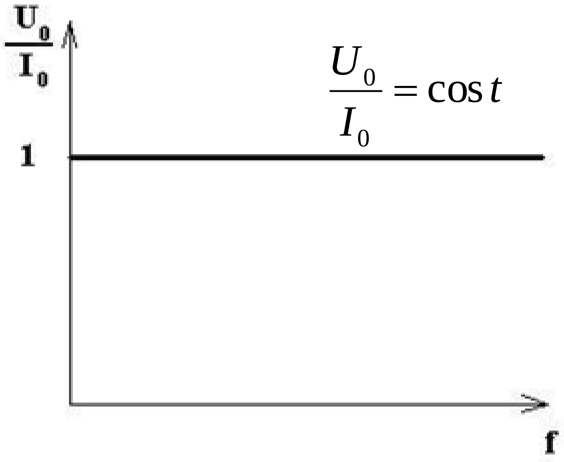
Frequency Response

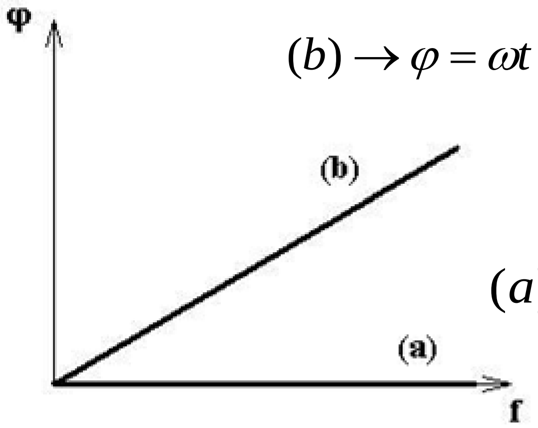
In the more general case, an input variable (*measurand*) changes “periodically” during time, then we can refer to simple sinusoidal inputs because every periodic signal can be decomposed by the ***Fourier Series***:

$$f(x) = f(x+2\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n sen(nx)\right)$$

input: $i(t) = I_0 sen\omega t$ with $\omega = 2\pi f$ ***frequency***

output: $u(t) = U_0 sen(\omega t + \varphi)$ with φ ***phase delay***



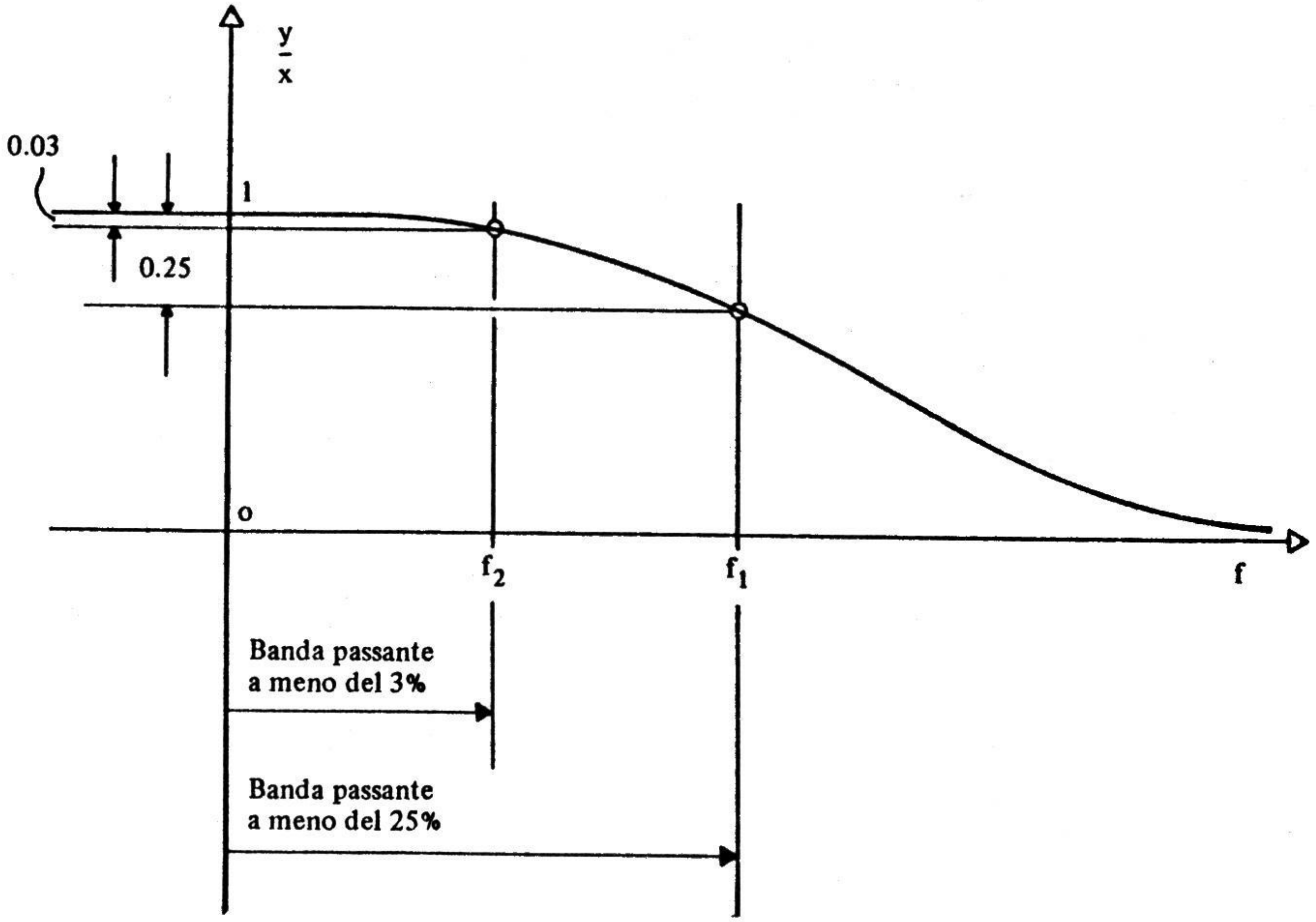


$$(b) \rightarrow \varphi = \omega t \rightarrow t_r = \frac{\varphi}{\omega} = \frac{\varphi}{2\pi f}$$

$$(a) \rightarrow \varphi = 0$$

ideal ***frequency response*** and ***phase response*** curves …

- constant *output/input ratio* for “every frequency” of the input !
- zero *phase delay* for “every frequency” of the input !



Real instrument ***frequency response*** example:

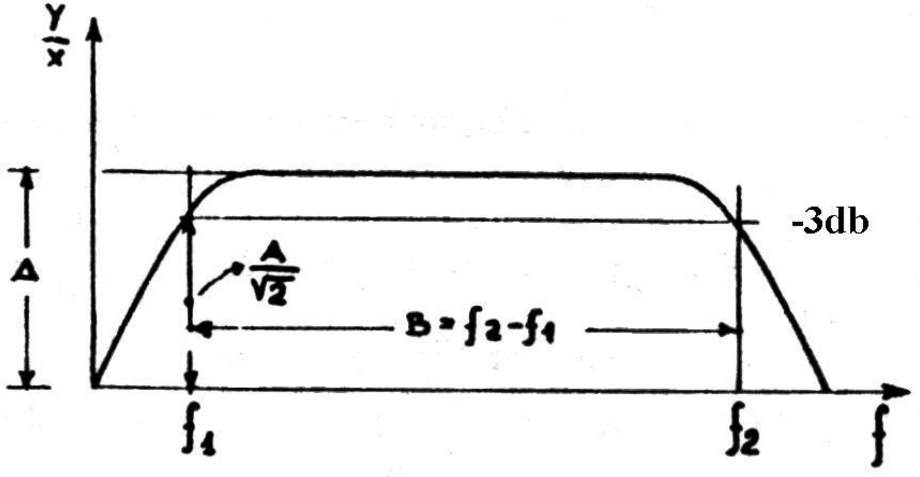
Real instruments always *attenuate the output amplitude* for higher frequencies (f → ∞) because:

- mechanical instrument have inner moving parts with *inertia* which can not have "infinite acceleration"
- electronic instrument have inner components which can not have *inductive reactance* $X_L = j\omega L = \infty$ *capacitive reactance* $X_C = 1/j\omega C = 0$

The extension of an *instrument frequency response* depends also from the ***dynamic error*** the operator accepts:

- if we accept a dynamic error ε_d = 3% we have a ***cut-off frequency*** f_2
- If we accept a dynamic error ε_d = 25% we have a much higher ***cut-off frequency*** $f_1 > f_2$

All frequencies between f_0 and f_1 (or f_2) form the ***instrument bandwidth or pass-band*** !



There are many instruments where the bandwidth does "not start" from frequency f_0 = 0 Hz, but starts at a higher value f_1
In these cases, we have "two cut-off frequencies" and the pass-band is between these two frequencies: $B = f_{ts} - f_{ti}$

Most of the times the *dynamic error* (which can be "fairly large") is expressed in a *logarithmic scal*e, inherited from "acustics" studies and the logarithmic unit is the ***decibel*** :

It is world wide accepted that a tolerable dynamic error limit is $-3dB = 20\log_{10} 0.707$ which means the instrument output ***A*** is "attenuating the input" to the 70,7 % of its real amplitude or that it is doing a ***dynamic error*** of 29,3 % !!

We change now notation for the *input* and *output variables:*

y(t) → <u>input variable</u> (***measurand***)
x(t) → <u>output variable</u> (***instrument response*** *or* ***indicator deflection***)

We say an instrument is <u>*dynamically linear*</u> if we can write its "dynamic characteristic equation" as follows :

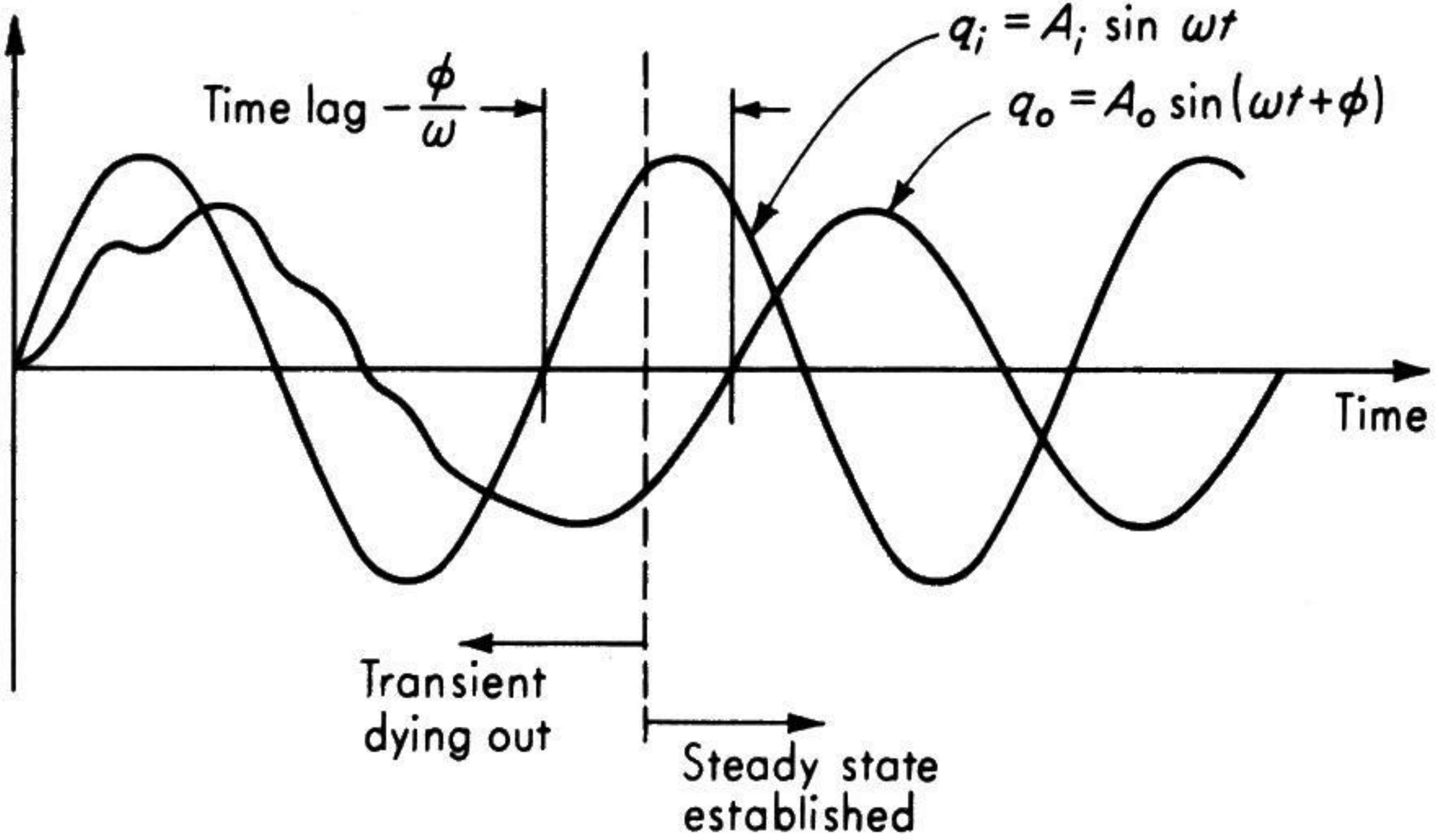
$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = y(t)$$

The instrument dynamic can be described by an <u>*ordinary differential equation*</u> with *constant coefficients* !
The solution can be written as:

$$x(t) = x_{tr}(t) + x_{rg}(t)$$

Solution of the *associated homogeneous equation* which represents: $x_{tr}(t)$
the <u>*transient dynamic response*</u> of the instrument

Solution of a *particular integral* of the equation which represents: $x_{rg}(t)$
the <u>*dynamic steady state*</u> of the instrument



To study the TWO different situations we will employ the TWO different experimental schemes or tools introduced before:

1. For the ***dying out transient*** → the **STEP RESPONSE** $x_{tr}(t)$
2. For the ***established steady state*** → the **FREQUENCY RESPONSE** $x_{rg}(t)$

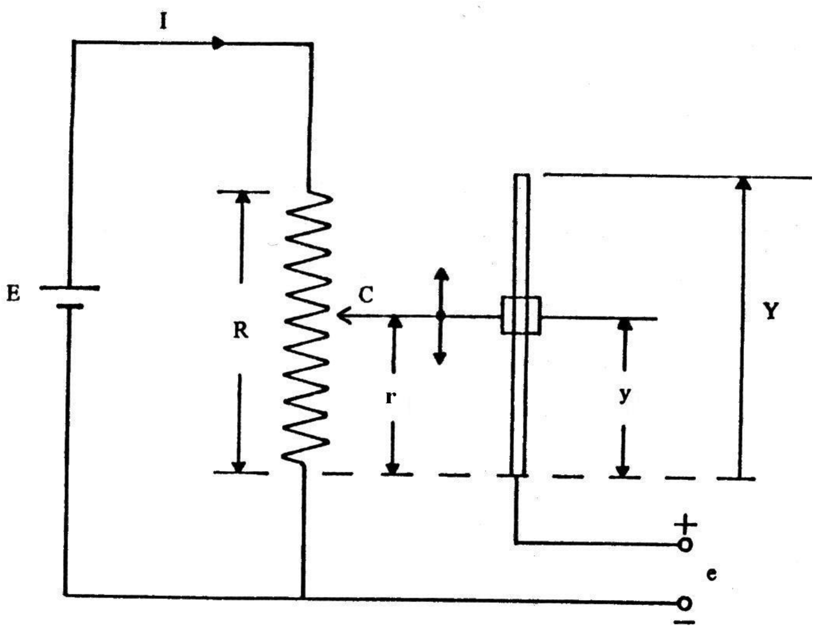
Zero-order instruments

When the differential equation of the dynamic characteristic has the simplest form, without the derivative terms:

$$a \cdot x = b \cdot y \qquad \rightarrow \qquad x = \frac{b}{a} y$$

The output signal (or deflection) is "always" proportional to the input signal ! There is no delay between output and input therefore, zero-order instruments "respond instantaneously" and can be considered *ideal instrument !*

Example:



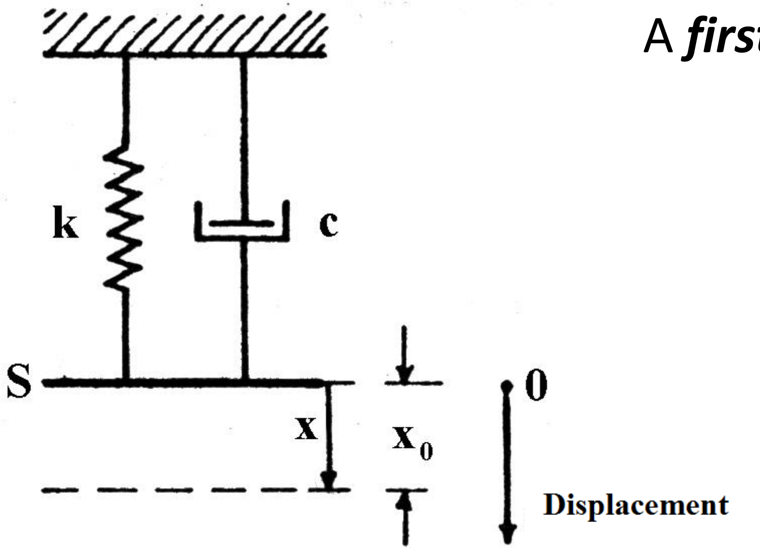
For the ***potentiometer*** we have for the left circuit *E = RI* while the displacement *y* is indicated by the output voltage *e(t)* which is:

$$e(t) = r(t)I = r(t)\frac{E}{R} = \frac{r(t)}{R}E = \frac{\rho \cdot y(t)/S}{\rho \cdot Y/S}E = \frac{y(t)}{Y}E$$

where $R = \rho \frac{Y}{S}$ and $r(t) = \rho \frac{y(t)}{S}$ the variable "*time*" is only present as "independent variable" for the position *y = y(t)*

Note that $e = \frac{E}{Y} y$ *is also the graduation curve* (***static characteristic curve***) *of the potentiometer and* $\frac{de}{dy} = \frac{E}{Y}$ *is the* ***static sensitivity !***

First-order instruments



A ***first-order instrument*** can "store energy" in only *one* of its inner elements !

If we displace the indicator *S* of the simple mechanical system to a position $x_0 \neq 0$, *elastic* (potential) *energy* is stored in the spring *k*.

If at $t = 0$ we leave the indicator *S* free of moving, the elastic energy stored in the spring will pull back the indicator to the rest position $x = 0$. During movement, the damper *c* will oppose the motion.

For every time instant *t* the equation of *force equilibrium* is: $kx = -c\dot{x}$ or $c\dot{x} + kx = 0$

For every time instant *t* the *velocity* is: $\dot{x} = -\frac{k}{c}x$ therefore the coefficient $\left[\frac{c}{k}\right] = [t]$ is a *time* !

$$\boxed{\frac{c}{k} = \lambda}$$

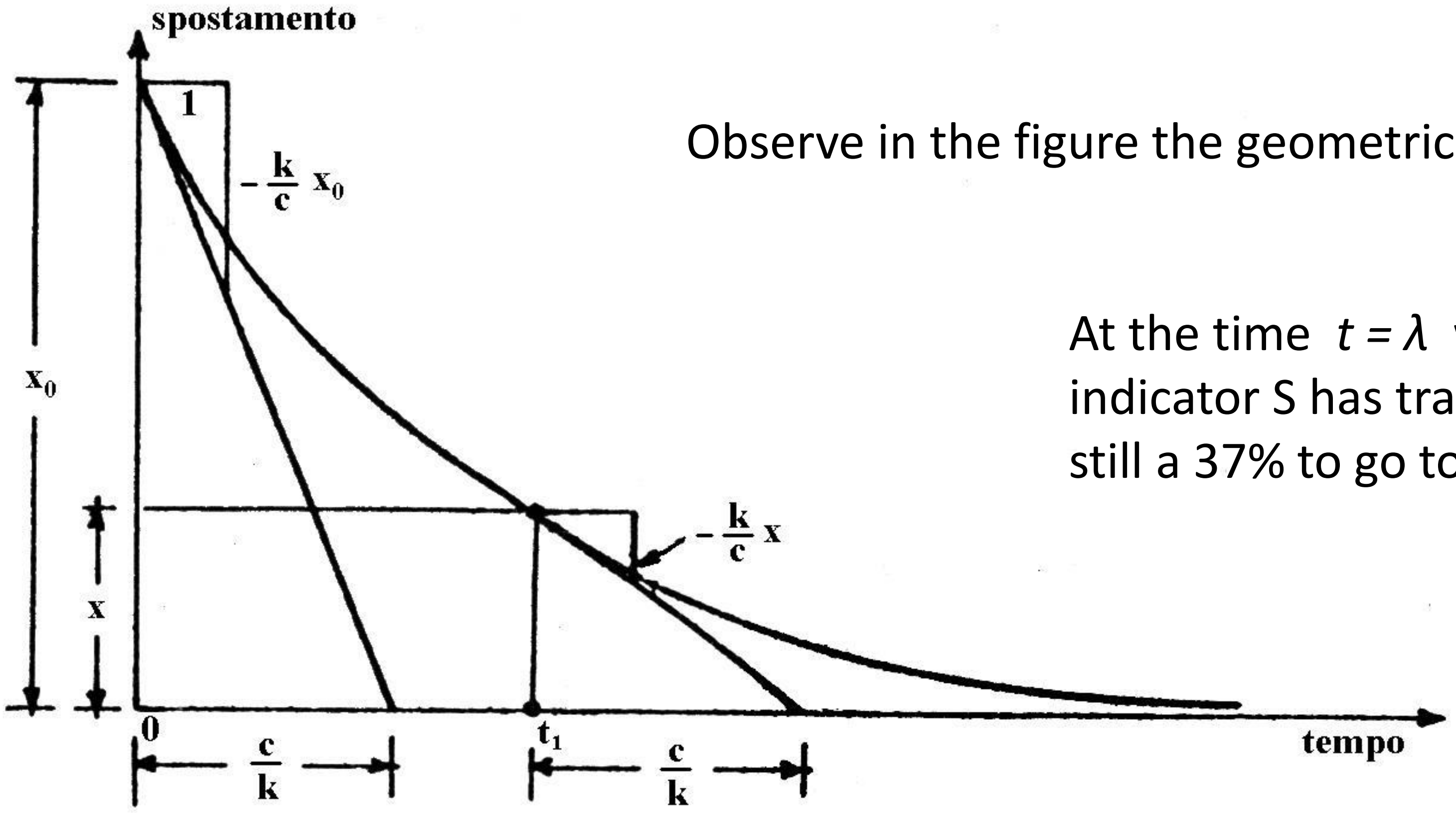
is the ***time constant*** of the first-order instrument of above !

The solution of the 1st *order differential equation* $c\dot{x}+kx=0$ is the well known "decreasing exponential":

$x(t)=x_0\cdot e^{-\frac{k}{c}t}$ the curve of which can be drawn with the *initial condition*: *t = 0* $\rightarrow$ $x(0)=x_0$

The tangent line to the curve in the point *t = 0* is: $\dot{x}(0)=-\frac{k}{c}x_0=-\frac{x_0}{c/k}$

Observe in the figure the geometrical meaning of the time constant $\lambda=\frac{c}{k}$



At the time *t = λ* we have that $x(\lambda)=x_0e^{-1}=0.37\cdot x_0$ the indicator S has travelled for 63% of the displacement and has still a 37% to go to reach its final position *x = 0* !!

… but these are only introductory definitions and considerations …

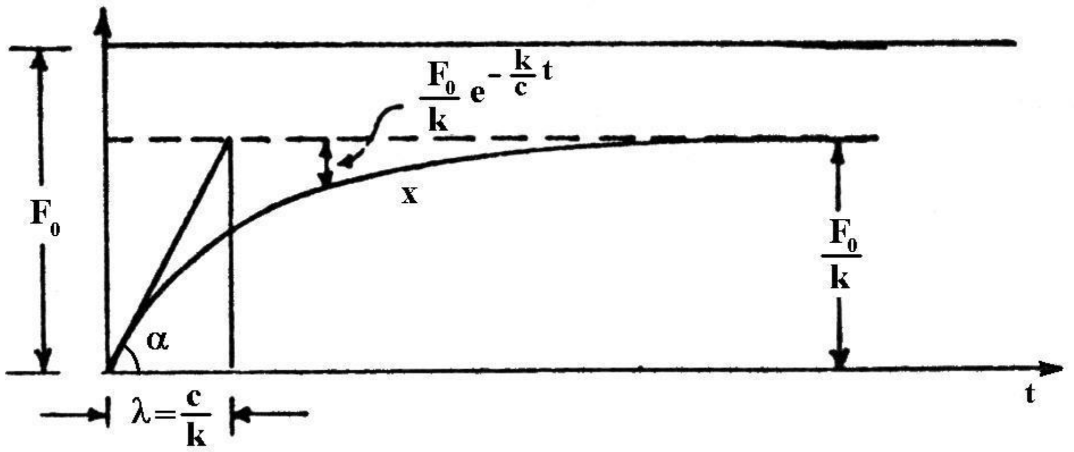
In the general case of the ***step response*** for the instrument of before, an *input variable* is present :
a step force F_0 applied to the indicator *S* at time *t = 0* …
The *1st order differential equation* that describes the indicator deflection therefore is: $c\dot{x}+kx=F_0$ and the general solution has the form $x(t)=x_{tr}+x_{rg}$

- x_{rg} is a *particular solution* and describes the "steady state condition" of the instrument:
  Note that when $t \to \infty$ the indicator speed must be zero $\dot{x}=0$ therefore the steady state of the indicator S will be described by the position $x_{rg}=F_0/k$

- x_{tr} instead, is the *solution of the associated homogeneous equation* and describes the "dying out transient":

$$x_{tr}=-\frac{F_0}{k}e^{-\frac{k}{c}t}$$

The final solution will be the "sum" of the two contributions: $x(t)=x_{tr}+x_{rg}=-\frac{F_0}{k}e^{-\frac{k}{c}t}+\frac{F_0}{k}=\frac{F_0}{k}\left(1-e^{-\frac{k}{c}t}\right)$

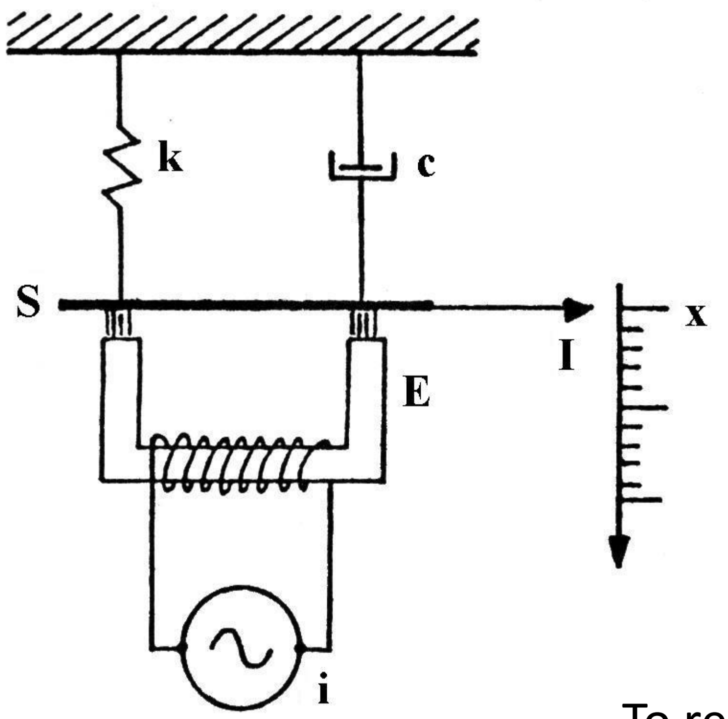


Observe again on the figure the meaning of $\lambda=\frac{c}{k}$

$$tg\alpha=\frac{F_0/k}{\lambda}=\dot{x}(0)$$

At $t=\lambda$ we have $x(\lambda)=\frac{F_0}{k}\left(1-e^{-1}\right)=0.63\frac{F_0}{k}$

When we are interests in studying the *steady state dynamic response* of a *1st order instrument* we have to employ the other tool, the ***frequency response***. To do so, we have to apply at the instrument input a *periodic measurand*:
$F(t) = F_0 sen\omega t$ a periodic force of frequency $\omega = 2\pi f$!



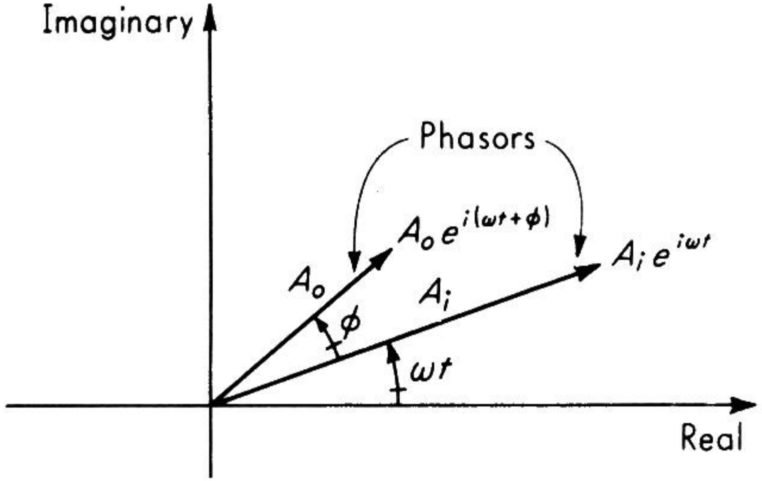
The differential equation which describes the indicator S movements will be: $c\dot{x} + kx = F_0 sen\omega t$
The general solution will always be $x(t) = x_{tr} + x_{rg}$ but we will be interested only in the steady state part, which we can assume to be:

$$x_{rg}(t) = X_0 sen(\omega t + \varphi)$$

How does the "amplitude" of the *output indication x(t)* change with frequency ? And what "delay" will the *output x(t)* show ?

To respond to those questions we will employ the <u>*phasor notation*</u>:

input: $F(t) = F_0 sen\omega t = F_0 e^{j\omega t}$
output: $x(t) = X_0 sen(\omega t + \varphi) = X_0 e^{j\omega t} e^{j\varphi}$



To verify the x_{rg} solution we have to substitute our assumed “steady state solution” in the differential equation …

$$j\omega c X_0 e^{j\omega t} e^{j\varphi} + k X_0 e^{j\omega t} e^{j\varphi} = F_0 e^{j\omega t}$$

$$X_0 e^{j\varphi}\left(j\omega c + k\right) = F_0$$

$$X_0 e^{j\varphi} = \frac{F_0}{j\omega c + k} = \frac{\frac{F_0}{k}}{j\omega \frac{c}{k} + 1}$$

where $\frac{F_0}{k}$ is the *maximum amplitude* and $\frac{c}{k} = \lambda$ is the ***time constant*** !

… calculating the modulus of the rationalized function, we will obtain:

$$X_0 = \frac{\frac{F_0}{k}}{\sqrt{(\omega\lambda)^2 + 1}}$$

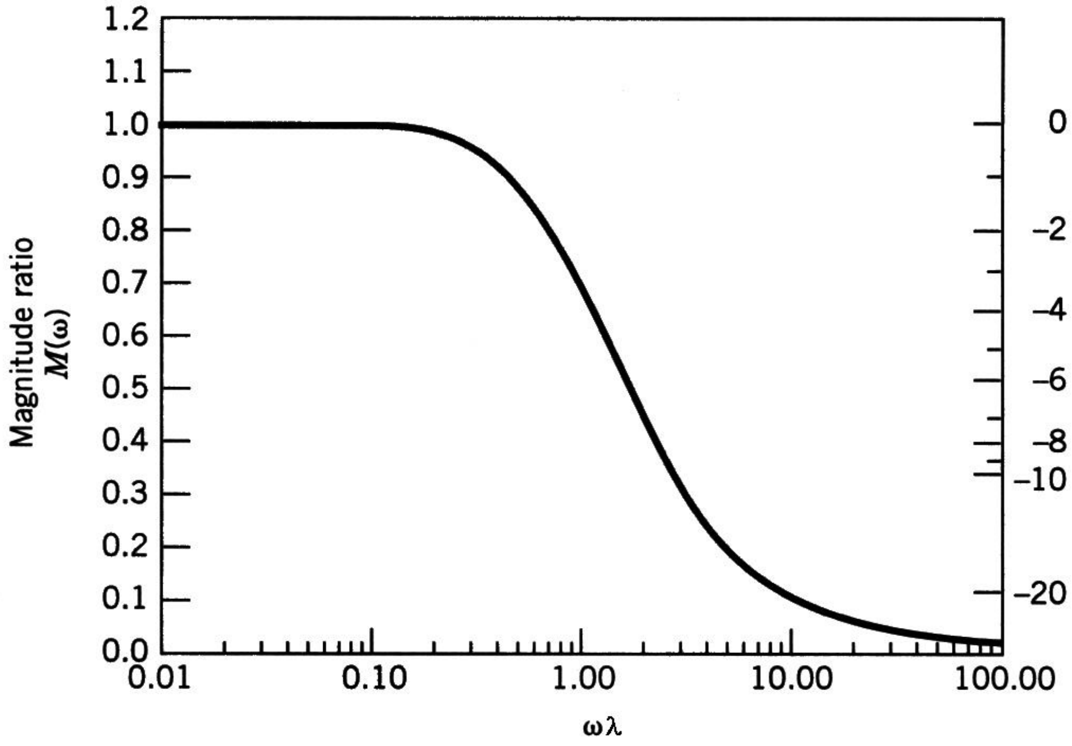
$$\boxed{\frac{X_0}{\frac{F_0}{k}} = G = \frac{1}{\sqrt{(\omega\lambda)^2 + 1}}}$$

which is the ***amplification*** or ***gain*** of the instrument !

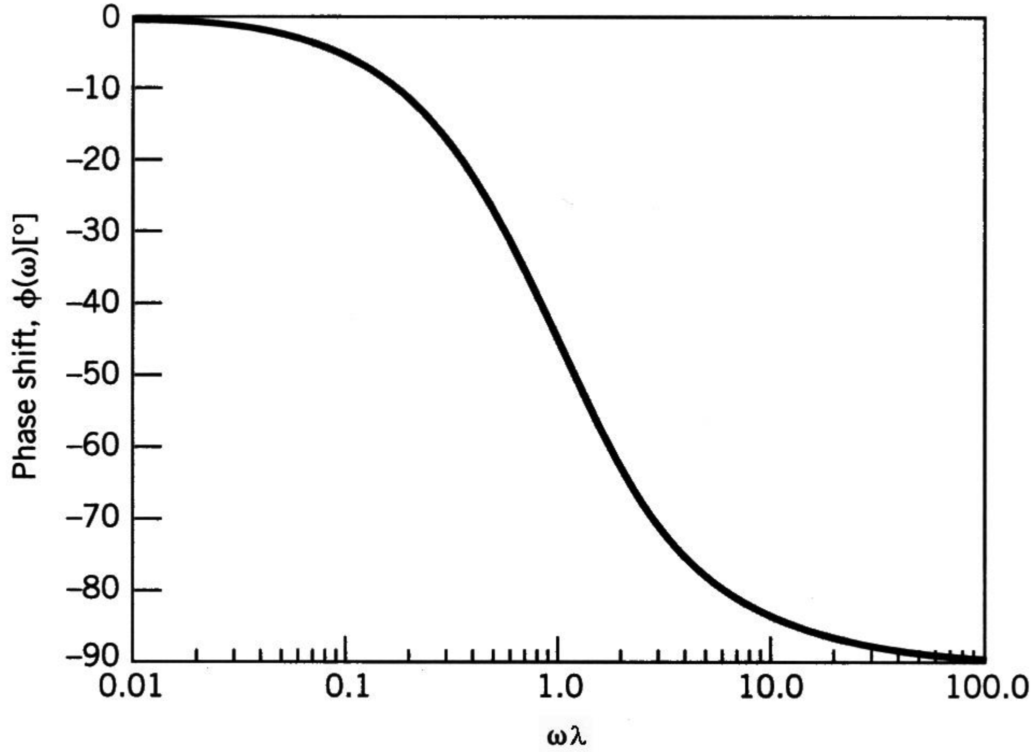
$$\boxed{\varphi = arctg\left(-\omega\lambda\right)}$$

which is the ***phase delay*** of the instrument !

1st order instrument <u>frequency response</u>



1st order instrument <u>phase delay</u>



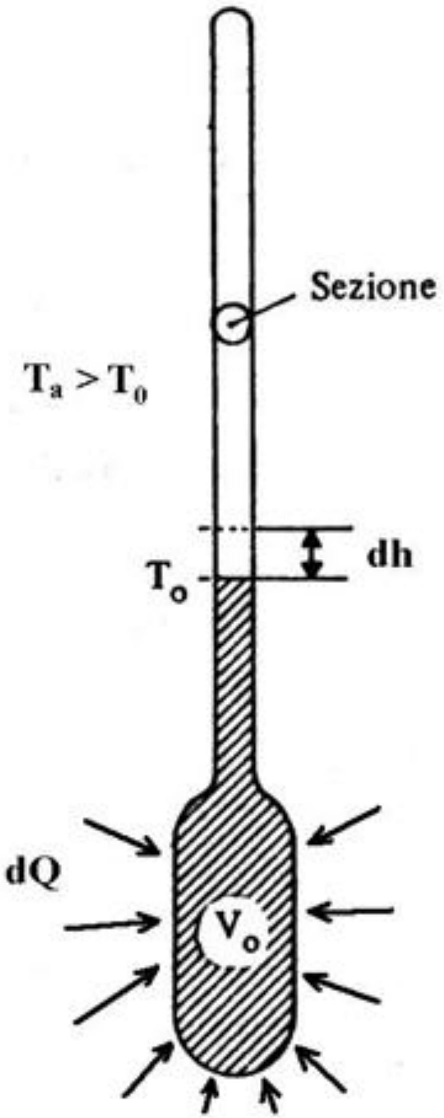
These diagram have on the *x* axis the non-dimensional variable *"ωλ»* and therefore are *<u>normalized diagrams</u>*

$\omega_c = \dfrac{1}{\lambda}$ is the ***characteristic frequency*** of a *1st order instrument* for which we have *$\omega_c\lambda$ = 1* → $G = \dfrac{1}{\sqrt{2}} \cong 0.707$

therefore, ω_c is the ***cut-off frequency*** for a - 3db attenuation : $f_c = \dfrac{1}{2\pi \cdot \lambda}$

Example:

The "dynamic response" of the mercury thermometer



For every *differential time interval dt* that precedes the equilibrium condition we have:

Heat amount the environment "transfers to" the mercury: $dQ = kA(T_a - T)dt$

Heat the mercury "receives from" the environment: $dQ = mc \cdot dT$

Where: *m* is the total mercury *mass*
c is the mercury *specific heat*
k is the *thermal exchange coefficient*
A is the *thermal exchange surface* of the thermometer

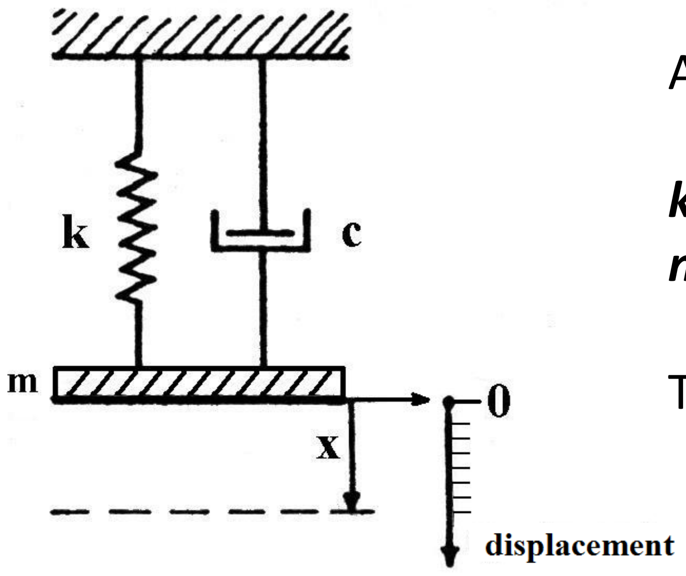
The two *heat amounts* of above must be the same:

$mc \cdot dT = kA(T_a - T)dt$ $\qquad mc \cdot \frac{dT}{dt} = kAT_a - kAT$ $\qquad \boxed{\frac{mc}{kA} \cdot \frac{dT}{dt} + T = T_a}$

$\boxed{\lambda = \frac{mc}{kA}}$ is the ***time constant*** for this 1st order instrument !

If $\lambda = 1s$ then $\omega\lambda = 1$ → $\omega_c \cdot 1 = 1$ and $2\pi f_c = 1$ → $f_c = \frac{1}{2\pi} = 0.16\,Hz$ *quite slow* !

Second-order instruments



A ***second-order instrument*** can "store energy" in <u>*two*</u> of its inner elements !

k → the elastic spring stores <u>*potential energy*</u>
m → the moving mass stores <u>*kinetic energy*</u>

The *differential equation* that describes the "free movement" of the indicator ***m***

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

If we consider the very particular case for which $c = 0$ we get the important limit situation described by :

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$$ where $$\boxed{\frac{k}{m}=\omega_n^2}$$ and $\boldsymbol{\omega_n}$ is the <u>***natural frequency***</u> of the instrument !

A system like this has NO damping and, if pulled from its equilibrium position $x = 0$, it will oscillate indefinitely with a frequency ω_n and will NEVER stop !
Of course, it's an IDEAL system …

In the real case of the ***step response*** for the instrument of before, an *input variable* is present :
a step force F_0 applied to the indicator *m* at time *t = 0* …
The 2^{nd} *order differential equation* that describes the indicator deflection therefore is: $m\ddot{x} + c\dot{x} + kx = F_0$ and the general solution has again the form $x(t) = x_{tr} + x_{rg}$

- x_{rg} is a *particular solution* and describes the "steady state condition" of the instrument:
  Note that when $t \to \infty$ the indicator speed and acceleration must be zero $\ddot{x} = \dot{x} = 0$ therefore the steady state of the indicator S will be described by the position $x_{rg} = F_0 / k$

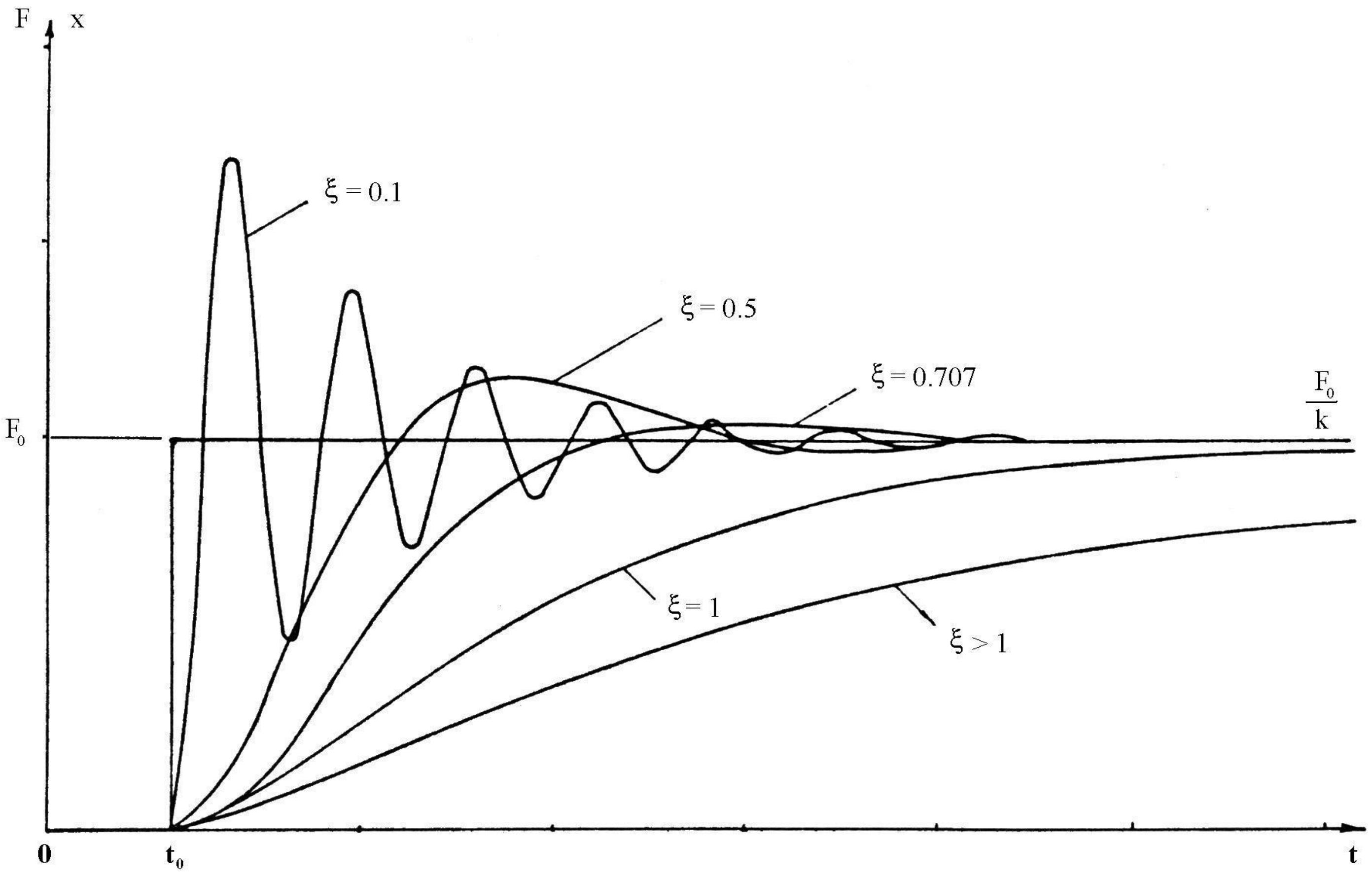
- x_{tr} instead, is the *solution of the associated homogeneous equation* and describes the "dying out transient":

  $x_{tr}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$ with $\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$ and $\sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} = \sqrt{\Delta}$ the ***discriminant*** !

  Depending on the sign of the discriminant $\Delta > 0$, $\Delta = 0$, $\Delta < 0$ we will have "different behaviors" of the instrument. Therefore, the 2^{nd} order instrument will exhibit "more than one type" of step response !
  The curves of the step response will depend on a parameter obtained from the discriminant Δ :

$$\xi = \sqrt{\frac{\frac{c^2}{4m^2}}{\frac{k}{m}}} = \sqrt{\frac{c^2}{4km}} = \frac{c}{2\sqrt{km}}$$

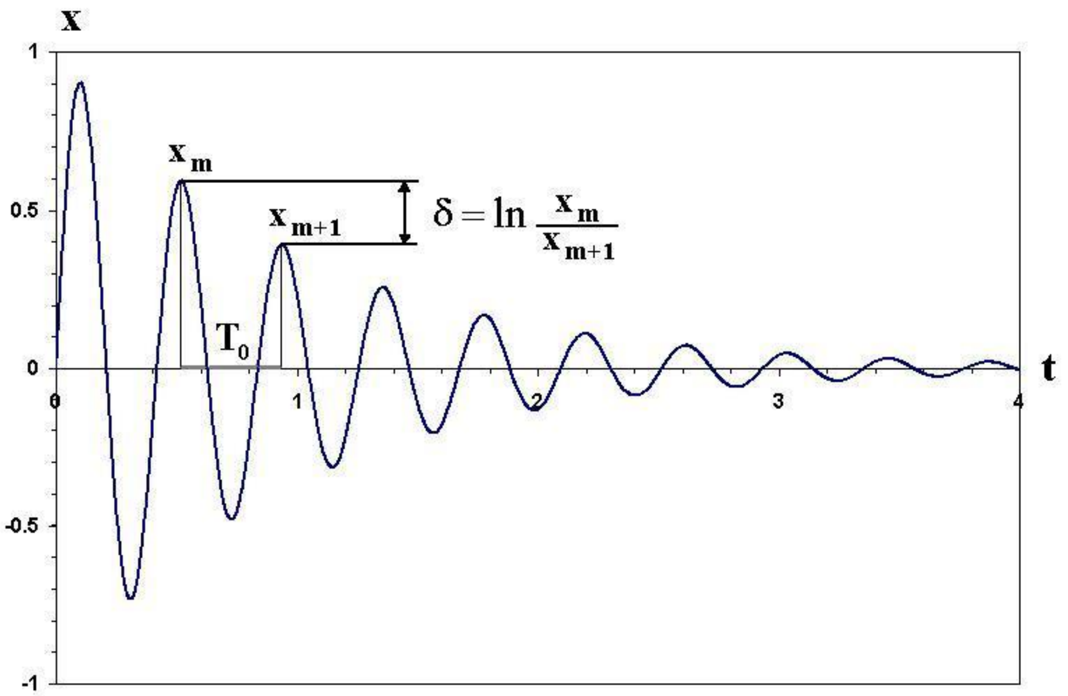
the ***damping factor*** $\boxed{\xi = \frac{c}{c_{cr}}}$ where $c_{cr} = 2\sqrt{km}$



- For $\xi > 1$ (which is the same of $c > c_{cr}$) the instrument is over-damped and it will reach the final indication with a slow exponential path !
- For $\xi = 1$ (which is $c = c_{cr}$) the instrument has a critical damping and reaches the final indication with the fastest exponential path !
- For $\xi < 1$ (which is $c < c_{cr}$) the instrument is under-damped, it reacts quickly to the step input but exhibits several damped *over-oscillations* before reaching the final indication !

Note that for the case $\xi < 1$ the frequency of the over-oscillations seems to depend from the value of the damping factor ξ …
Instrument designer generally prefer to choose a damping factor $\xi = 0.7 \div 0.8$ so the indicator ***m*** will have only one over-oscillation and reaches the final indication F_0/k from the upper side …

To obtain the values of $\boldsymbol{\omega_n}$ and $\boldsymbol{\xi}$ one should know the values of the coefficients ***m***, ***c***, and ***k*** of the 2nd order instrument which is often almost impossible !
In case the instrument has $\boldsymbol{\xi < 1}$ *(which is the most common)* one can employ an experimental procedure that leads to the ***logarithmic decay δ*** :



Displace the instrument indicator from the zero position, leave it free to move back and measure the decreasing oscillations …

$$\delta = \ln \frac{x_m}{x_{m+1}} = \frac{1}{k} \ln \frac{x_m}{x_{m+k}}$$

If the decay is small, it can also be measured "after *k* waves", as showed in the equation above.

The *logarithmic decay* **δ** is dependent from the *damping factor* **ξ** : $\delta = \ln \frac{A_n}{A_{n+1}} = 2\pi \frac{\xi}{\sqrt{1-\xi^2}}$

And the frequency $\omega_0 = 2\pi/T_0$ of the decreasing oscillation is "smaller" than the natural frequency $\boldsymbol{\omega_n}$ … depending also from the damping factor ***δ*** : $\omega_0 = \omega_n \sqrt{1-\xi^2}$

When we are interests in studying the *steady state dynamic response* of a *2nd order instrument* we have to employ the other tool, the ***frequency response***. To do so, we have to apply at the instrument input a *periodic measurand*: $F(t) = F_0 sen\omega t$ a periodic force of frequency $\omega = 2\pi f$!

The differential equation which describes the indicator ***m*** movements will be: $m\ddot{x} + c\dot{x} + kx = F_0 sen\omega t$
The general solution will always be $x(t) = x_{tr} + x_{rg}$ but we will be interested only in the steady state part, which we can assume to be: $x_{rg}(t) = X_0 sen(\omega t + \varphi)$

How does the “amplitude” X_0 of the *output indication x(t)* change with frequency ?
And what “phase delay” φ will the *output x(t)* show ?

Same as done for the 1st order instruments we substitute the input $F(t) = F_0 e^{j\omega t}$ and the instrument output $x(t) = X_0 e^{j\omega t} e^{j\varphi}$ in the differential equation:

$$m(-\omega^2) X_0 e^{j\omega t} e^{j\varphi} + c(j\omega) X_0 e^{j\omega t} e^{j\varphi} + k X_0 e^{j\omega t} e^{j\varphi} = F_0 e^{j\omega t}$$

$$X_0 e^{j\varphi}(-m\omega^2 + jc\omega + k) = F_0 \qquad X_0 e^{j\varphi} = \frac{F_0}{-m\omega^2 + jc\omega + k} = \frac{\frac{F_0}{k}}{-\frac{m}{k}\omega^2 + j\frac{c}{k}\omega + 1}$$

with $\frac{k}{m} = \omega_n^2$ and $j\frac{c}{k}\omega = j\frac{\xi c_{cr}}{k}\omega = j\frac{\xi 2\sqrt{km}}{k}\omega = j2\xi\sqrt{\frac{m}{k}}\omega = j2\xi\frac{\omega}{\omega_n}$

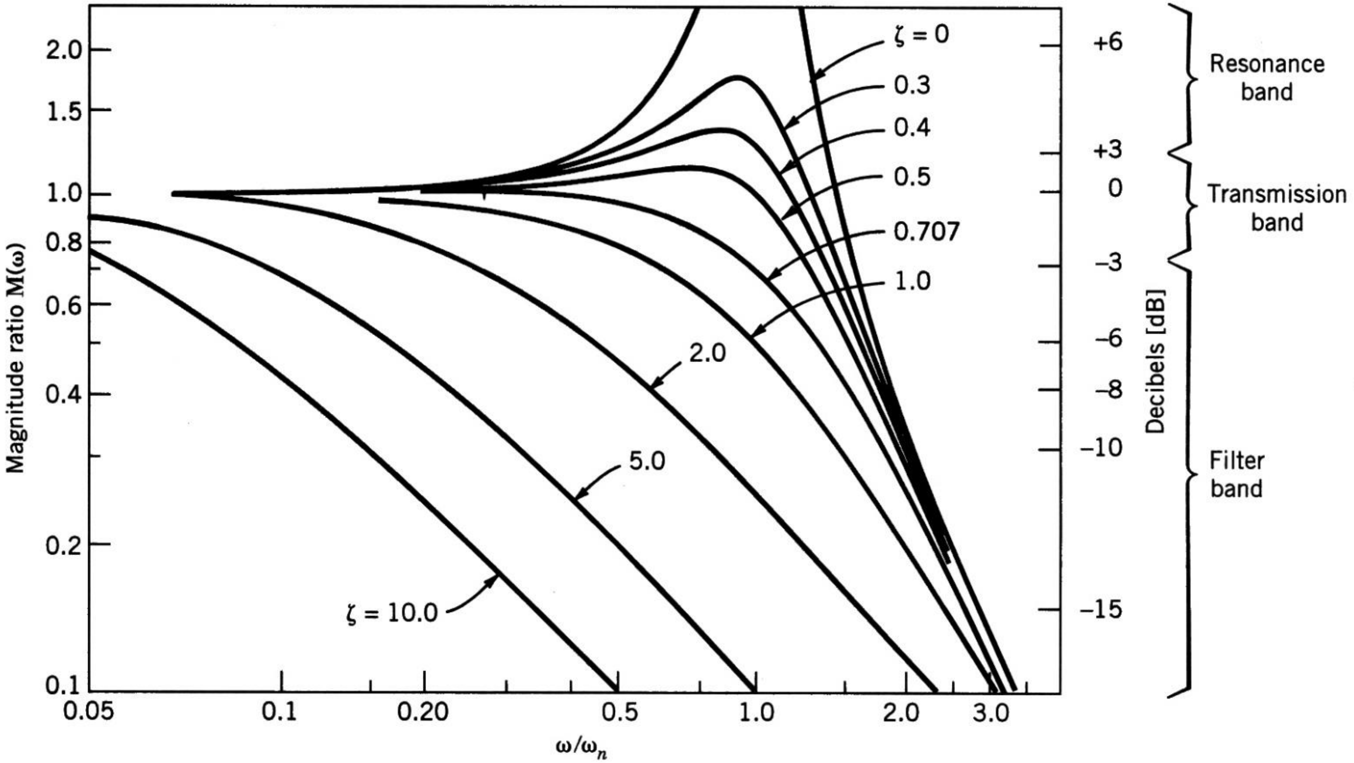
$$X_0 e^{j\varphi} = \frac{\frac{F_0}{k}}{-\frac{\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1} = \frac{\frac{F_0}{k}}{1 - \frac{\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n}}$$

which is the *complex frequency response* for the 2nd order instrument

Calculating the *modulus* and the *phase lag* of this function we obtain …

$$X_0 = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

the ***amplitude*** of the ***frequency response*** of the instrument !



Normalized diagram of the ***amplification*** or ***gain*** of the instrument:

$$G = \frac{X_0}{F_0 / k}$$

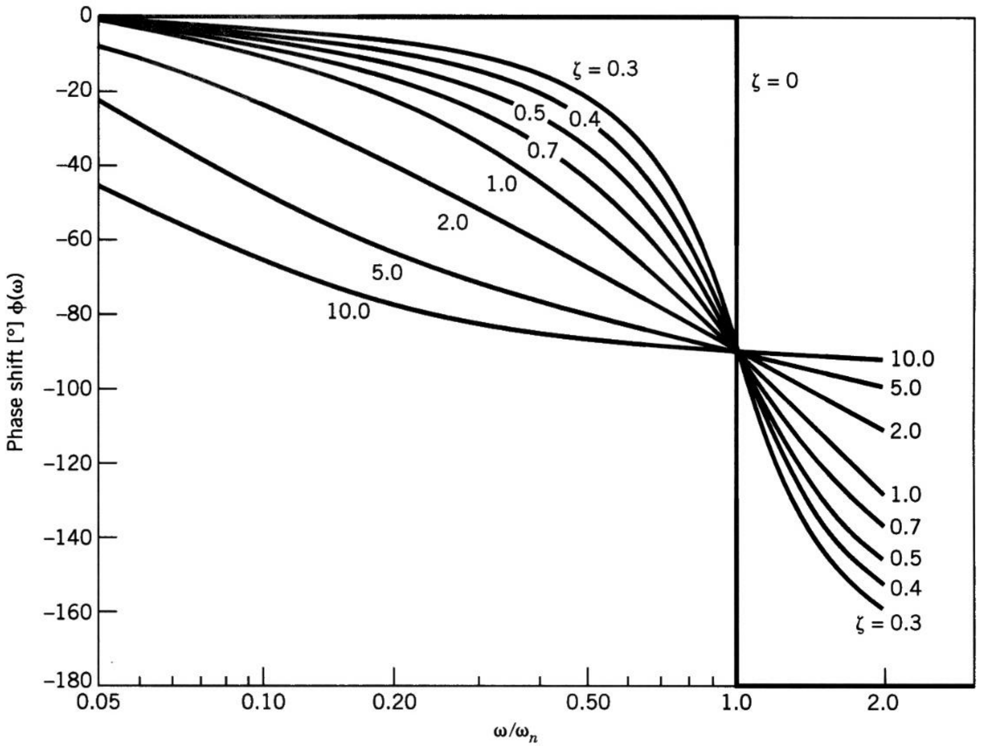
These curves are also dependent from the damping factor ξ

For $\frac{\omega}{\omega_n} = 1$ the amplification is: $G = \frac{1}{2\xi}$

wich for «small values» of ξ can be «dangerous», these are the ***resonance conditions*** …

$$\varphi = arctg \frac{-2\xi \frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

the ***phase delay*** of the instrument output !



The curves depend on the damping factor ξ , however they all pass through the point :

$$\frac{\omega}{\omega_n} = 1 \qquad \varphi = -\frac{\pi}{2}$$

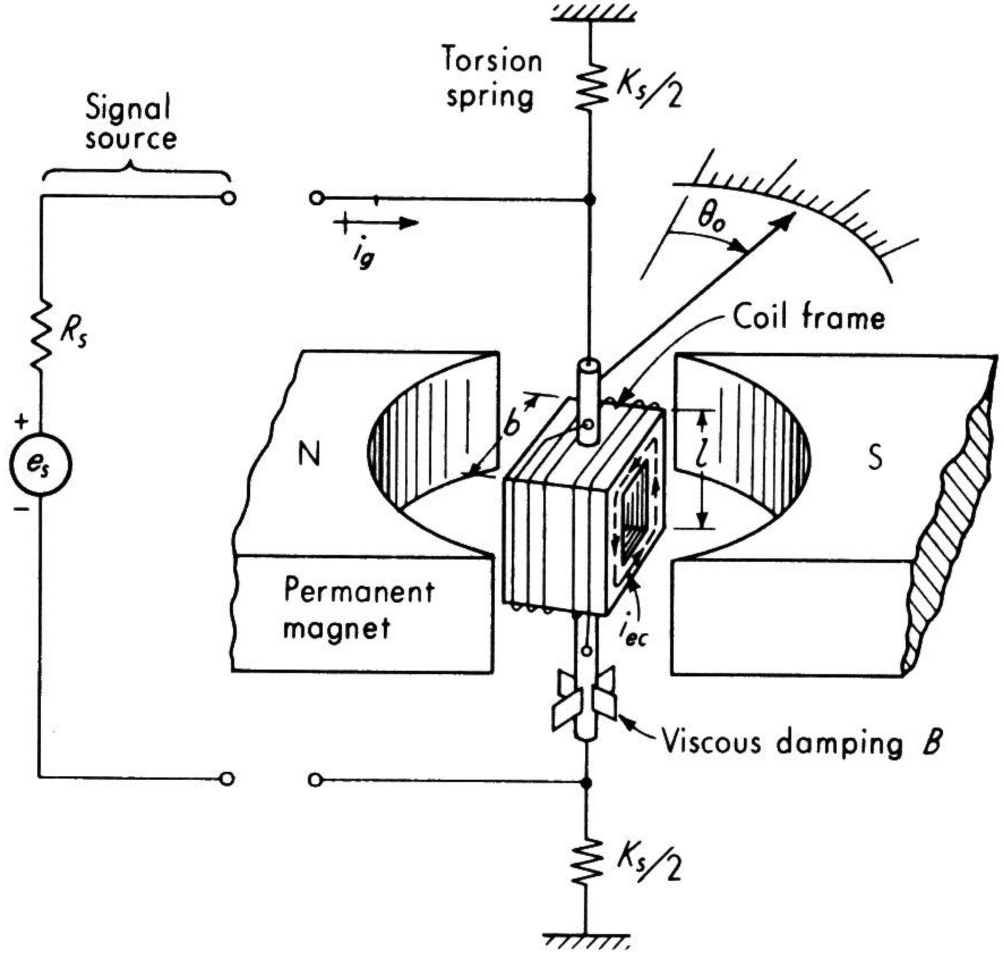
which makes it useful to experimentally determine the natural frequency ω_n without knowing the damping factor ξ …

If you consider $x_{rg}(t) = \frac{F_0}{k}$ the steady state output for the step response, the static sensitivity will result:

$$S = \frac{du}{di} = \frac{dx}{dF} = \frac{1}{k} = \frac{1}{m\omega_n^2}$$

which makes the ***dynamic response*** of an instrument *always inversely proportional* to its ***static sensitivity*** !!

Example: the galvanometer



Static characteristic (graduation curve)

$$\vec{F} = i\vec{l} \times \vec{B} \qquad |F| = ilB$$

motor torque: $C_m = nF \cdot b$

resisting torque: $C_r = k \cdot \theta$

$$C_m = C_r \qquad n \cdot ilB \cdot b = k\theta \qquad \boxed{\theta = \frac{nlBb}{k} \cdot i}$$

$$S = \frac{d\theta}{di} = \frac{nlBb}{k} = \frac{nlBb}{J\omega_n^2}$$

the sensitivity is constant and inversely proportional to ω_n

Dynamic characteristic (dynamic response)

The output indication is a rotation θ therefore : $\omega_n = \sqrt{\dfrac{k}{J}}$

$$\boxed{J\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = C(t)}$$

and $C(t) = C_0$ for t > 0 (step response)

$C(t) = C_0 sen\omega t$ (frequency response)